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Convergence Conditions for Vector Stieltjes Continued Fractions[1]

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Necessary and sufficient conditions for the convergence of vector S-fractions are obtained, generalizing classical results of Stieltjes. A class of unbounded difference operators of high order possessing a set of spectral measures is described.

1. INTRODUCTION

A classical theorem of Stieltjes (see [10]) gives a solution to the problem of the uniform convergence of the continued fraction

$$S(z) = \cfrac{1}{b_1 z + \cfrac{1}{b_2 + \cfrac{1}{b_3 z + \ddots + \cfrac{1}{b_{2k-1} z + \cfrac{1}{b_{2k} + \ddots}}}}} = \frac{1|}{|b_1 z} + \frac{1|}{|b_2} + \frac{1|}{|b_3 z} + \cdots, \tag{1}$$

where $b_{2k-1} < 0$ and $b_{2k} > 0$, $k \in \mathbb{N}$, on each compact subset of $\mathbb{C} \setminus [0, +\infty)$. Writing S_n for the nth convergent of $S(z)$, one has

$$\sum_{k=1}^{\infty} |b_k| = \infty \Leftrightarrow S_n(z) \underset{n \to \infty}{\rightrightarrows} S(z),$$
$$z \in K \subseteq \mathbb{C} \setminus [0, +\infty). \tag{2}$$

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The S-fraction (1) can be written in the equivalent form

$$S(z) = \frac{a_0|}{|z} + \frac{-a_1|}{|1} + \frac{-a_2|}{|z} + \cdots, \frac{-a_{2k-1}|}{|1} + \frac{-a_{2k}|}{|z} + \cdots,$$

where

$$a_k = -\frac{1}{b_k b_{k+1}}, \qquad a_0 = \frac{1}{b_1}. \tag{3}$$

The Stieltjes continued fraction gives a formal expansion of the resolvent function f (Weyl's function) of the second order difference operator A, which is the closure of the operator given by the tridiagonal matrix:

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & \\ a_1 & 0 & 1 & 0 & \cdots & & \\ 0 & a_2 & 0 & 1 & \cdots & & \\ 0 & 0 & a_3 & 0 & 1 & \cdots & \\ 0 & 0 & \ddots & & & & \ddots \end{pmatrix}, \qquad a_k > 0, \tag{4}$$

via the usual matrix product on the linear subspace C_0 of the Hilbert space $l_2(\mathbb{N})$, formed by the finite linear combinations of its standard basis elements $e_0, e_1, \ldots$. The operator A is the general prototype of a second order difference operator. Assuming without loss of generality $a_0 = 1$, for the Weyl function

$$f(z) = \langle (zI - A)^{-1}\, e_0, e_0 \rangle = \sum_{k=0}^{\infty} \frac{\langle A^k e_0, e_0 \rangle}{z^{k+1}} = \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}}$$

we have the relations

$$\begin{aligned} f(z) &= zS(z^2), \qquad & z &\notin (-\infty, +\infty), \\ f_{2k+1} &= 0, \qquad & k &\geqslant 0. \end{aligned}$$

The convergence conditions of the Stieltjes functions, in terms of the entries of the operator (4) or those of the continued fraction (1), are very important in order to obtain the spectral properties of this operator, especially when the operator is not bounded. The convergence of the Stieltjes continued fraction is equivalent to the unicity of the spectral measure of the operator (4), and at the same time to the existence of a unique self-adjoint extension for A. Taking into account equality (3), the divergence of the series in (2) obviously gives a useful sufficient condition for the convergence of (1), which can be written as

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty \Rightarrow S_n(z) \underset{n\to\infty}{\rightrightarrows} S(z),$$
$$z \in K \subseteq \mathbb{C} \backslash [0, +\infty). \tag{5}$$

2. THE VECTOR CASE

In order to obtain the spectral properties of the difference operator A, of order $p+1$, given as the closure of the operator defined on $C_0 \subset l_2(\mathbb{N})$ by the infinite matrix

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & & & \\ 0 & 0 & 1 & 0 & \cdots & & & & \\ \cdots & \cdots & \cdots & & & & & & \\ 0 & & & 0 & 1 & & & & \\ a_1 & 0 & \cdots & & 0 & 1 & & & \\ 0 & a_2 & 0 & \cdots & & 0 & 1 & & \\ 0 & 0 & \ddots & & & & & \ddots & \\ \cdots & \cdots & \cdots & & & & & & \end{pmatrix}, \qquad a_k \in \mathbb{C}, \tag{6}$$

vector generalizations of the Stieltjes fractions were given in [3],

$$\begin{aligned} \vec{\mathscr{S}}(z) &= (S^1(z), \ldots, S^p(z)) \\ &= \frac{(1, \ldots, 1, a_0)|}{|(0, \ldots, 0, z)+} \frac{(1, \ldots, 1, -a_1)|}{|(0, \ldots, 0, 1)+} \cdots \frac{(1, \ldots, 1, -a_p)|}{|(0, \ldots, 0, 1)+} \\ &\quad \times \frac{(1, \ldots, 1, -a_{p+1})|}{|(0, \ldots, 0, z)+ \cdots}, \end{aligned} \tag{7}$$

where, according to the Jacobi–Perron algorithm, the product and quotient of two vectors a, b of $\mathbb{C}^p$ are defined by the formulas

$$a \cdot b = (a_1 b_1, a_2 b_2, \ldots, a_p b_p),$$
$$\frac{1}{a} = \left(\frac{1}{a_p}, \frac{a_1}{a_p}, \ldots, \frac{a_{p-1}}{a_p}\right).$$

The n-convergent of $\vec{S}$ will be denoted by $\overrightarrow{S_n}$. Analogously to the scalar case, the set of resolvent functions of the operator A,

$$\vec{f} = (f^1, ..., f^p),$$

where

$$f^j(z) = \langle (zI - A)^{-1} e_{j-1}, e_0 \rangle = \sum_{k=0}^{\infty} \frac{f_{k,j}}{z^{k+1}}, \qquad j = 1, ..., p,$$

which define the operators in a unique way, can be expanded in a vector continued fraction (7). In this case, assuming $a_0 = 1$, we have the relations

$$f^j(z) = z^{p-j+1} S^j(z^{p+1}), \qquad j = 1, ..., p.$$

Bounded operators were considered in [3]. In particular, it was proved that the condition

$$0 < a_i < c, \qquad i \in \mathbb{N} \tag{8}$$

is sufficient for the resolvent functions f^j, $j = 1, ..., p$, to be Markov functions, represented as Cauchy transforms of measures supported on compact subsets contained in the positive semi-axes of the real line $[0, +\infty)$. The bounded case was also considered in [9]. Taking into account the previous assertions, it is easy to conclude that condition (8) implies the uniform convergence of each component of the continued fraction (7) on every compact subset of the complement of some compact set $\Gamma \subset \mathbb{R}_+$ in the complex plane:

$$\overrightarrow{S_n}(z) \underset{n \to \infty}{\rightrightarrows} \vec{S}(z), \qquad z \in K \subseteq \mathbb{C} \setminus \Gamma.$$

3. CONVERGENCE RESULTS

In the present paper, we aim to generalize Stieltjes' conditions for the convergence of the vector S-fraction (7). The announcement of the results of this paper has already appeared in [2]. Here we focus on the detailed proofs. We have the following theorem:

THEOREM 1. *If*

$$\sum_{k=1}^{\infty} \frac{1}{\sum_{j=0}^{p-1} a_{k+j}} = \infty, \qquad a_k > 0, \tag{9}$$

then $\overrightarrow{S_n} \underset{n \to \infty}{\rightrightarrows} \vec{S}$ *on each compact subset of* $\mathbb{C} \setminus [0, \infty)$.

Proof of Theorem 1. For a given j, $1 \leqslant j \leqslant p$, let S_n^j be the jth component of $\overrightarrow{S_n}$. In order to prove Theorem 1, according to Vitali's theorem (see [6, p. 341]) it is enough to show that $(S_n^j)_{n \geqslant 0}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus [0, +\infty)$ and that

$$S_n^j(x) \underset{n \to \infty}{\longrightarrow} S^j(x), \qquad x \in (-\infty, 0).$$

The proof of the fact that the convergents of the fraction (7) form a normal family in $\mathbb{C} \setminus [0, \infty)$ follows directly from the results of [3, Sect. 6.1]. For the sake of completeness we will present here some details. We write

$$S_n^j(z) = \frac{A_{n,j}}{A_{n,0}}(z), \qquad 1 \leqslant j \leqslant p.$$

The polynomials $A_{n,j}(z)$, for all $j \in \{0, ..., p\}$, as denominator and numerators respectively of a vector continued fraction, satisfy the recurrence relation (see [3, 4, 8])

$$A_{n+1,j} = \varepsilon_n A_{n,j} - a_{n-p+1} A_{n-p,j}, \qquad n \geqslant p$$

where

$$\varepsilon_n = \begin{cases} z, & n = k(p+1), \\ 1, & \text{otherwise.} \end{cases}$$

The initial conditions are

$$A_{0,0} = 1, \qquad A_{i,0} = z, \qquad A_{0,i} = 0,$$

and

$$A_{k,i} = \begin{cases} 1, & k \geqslant i, \\ 0, & \text{otherwise,} \end{cases} \qquad i = 1, ..., p.$$

It is clear from the previous relations that each $A_{n,j}$, $0 \leqslant j \leqslant p$, is a monic polynomial and that

$$\begin{aligned} &\deg A_{n,0} = \left[\frac{n+p}{p+1}\right], \\ &\deg A_{n,j} = \deg A_{n,0} - 1, \qquad j \in \{1, ..., p\}. \end{aligned} \tag{10}$$

For each n let $m = m(n)$ denote the degree of the common denominator $A_{n,0}$ of every component of $\overrightarrow{S_n}$ and $x_{i,m}$, $1 \leqslant i \leqslant m$, its zeros. In [3, Lemma 7] the following auxiliary results were proved:

- The zeros of $A_{k(p+1),j}$, $j=0, \ldots, p$, are simple and belong to $[0, +\infty)$;
- The zeros of $A_{k(p+1),0}$ and $A_{(k-1)(p+1),0}$ as well as the zeros of $A_{k(p+1),0}$ and $A_{k(p+1),j}$ (for any j between 1 and p) interlace.

These properties are also true for $n=k(p+1)+l$ with $l \in \{1, \ldots, p\}$. In fact, for each $x_{i,m}$, $1 \leqslant i \leqslant m$, one has (see [3, Lemma 5])

$$A_{k(p+1)+l,0}(x_{i,m}) \cdot A_{k(p+1),j}(x_{i,m}) < 0, \qquad l, j \in \{1, \ldots, p\}.$$

Taking into account the interlacing properties of the zeros of $A_{k(p+1),0}$ and those of $A_{k(p+1),j}$, we conclude that the zeros of $A_{k(p+1)+l,0}$ are simple and belong to $[0, +\infty)$.

Thus, for each fixed j one has

$$S_n^j(z) = \sum_{i=1}^{m} \frac{\mu_{i,m}^j}{z - x_{i,m}}, \qquad 1 \leqslant j \leqslant p, \quad m = m(n), \tag{11}$$

where $\mu_{i,m}^j > 0$. Indeed, from the interlacing properties of the zeros it is easy to check that

$$\mu_{i,m}^j = \frac{A_{n,j}}{A'_{n,0}}(x_{i,m}) > 0.$$

Let K be a fixed compact subset of $\mathbb{C} \setminus [0, \infty)$ and

$$\eta = \operatorname{dist}(K, [0, \infty)) = \min\{|z-\lambda| : z \in K, \lambda \in [0, \infty)\},$$

then we have $\eta > 0$. From (11) we obtain

$$|S_n^j(z)| \leqslant \frac{1}{\eta} \sum_{i=1}^{m} \mu_{i,m}^j, \qquad j=1, \ldots, p, m=m(n), \quad z \in K.$$

To see that $\{\sum_{i=1}^{m(n)} \mu_{i,m}^j\}_n$ is bounded we write

$$S_n^j(z) = \frac{c_{n,0}^j}{z} + \frac{c_{n,1}^j}{z^2} + \cdots,$$

where $c_{n,0}^j = \sum_{i=1}^{m(n)} \mu_{i,m}^j$. Then

$$c_{n,0}^j = \lim_{z \to \infty} z S_n^j(z),$$

and thus, considering (10), we obtain that

$$c_{n,0}^j = 1, \qquad n \in \mathbb{N}, j \in \{1, \ldots, p\},$$

from where one concludes that each component of $\vec{S_n})_{n \geqslant 0}$ is uniformly bounded on K. In order to analyze the pointwise convergence of every component of $\vec{S_n}(x) = (S_n^1(x), S_n^2(x), \ldots, S_n^p(x))$ for a fixed x on $(-\infty, 0)$, we put $S_n = S_n^1(x)$ and without loss of generality we restrict our attention to the convergence of the sequence $(S_n)_{n \geqslant 0}$. First we write (7) in an equivalent form

$$\vec{S}(z) = \cfrac{1}{(0, \ldots, 0, b_1 z) + \cfrac{1}{(0, \ldots, 0, b_2) + \ddots + \cfrac{1}{(0, \ldots, 0, b_{p+1}) + \cfrac{1}{(0, \ldots, 0, b_{p+2} z) + \ddots}}}}, \tag{12}$$

with

$$a_n = -\frac{1}{b_n b_{n+1} \cdots b_{n+p}}. \tag{13}$$

Notice that for $p = 1$ we obtain the classical Stieltjes fraction (1). For each fixed $x \in (-\infty, 0)$ we define the positive real numbers

$$b_n(x) = \begin{cases} b_n x, & \text{for } n \equiv 1 \pmod{p+1}, \\ b_n, & \text{otherwise.} \end{cases} \tag{14}$$

For shortness, in what follows we will write $b_n = b_n(x)$, so that $b_n > 0$ for every $n \in \mathbb{N}$. For the denominators $\Delta_n = \Delta_n(x)$ of S_n written in the new equivalent form (12), we have the recurrence relation

$$\Delta_n = b_n \Delta_{n-1} + \Delta_{n-p-1}, \tag{15}$$

with the initial conditions $\Delta_0 = 1$, $\Delta_1 = b_1$, $\Delta_2 = b_1 b_2$, ..., $\Delta_{p-1} = b_1 b_2 \cdots b_{p-1}$.

Writing

$$t_n = b_n \frac{\Delta_{n-1}}{\Delta_n}, \qquad n \geqslant 1, \tag{16}$$

we have that $t_1 = t_2 = \ldots = t_p = 1$ and

$$0 < t_n < 1, \qquad n > p. \tag{17}$$

From (15) we obtain the recurrence

$$S_n = t_n S_{n-1} + (1 - t_n)\, S_{n-p-1}, \tag{18}$$

with the corresponding initial conditions for $S_0, \ldots, S_p$. We define the sequences

$$M_n = \max\{S_n, S_{n-1}, \ldots, S_{n-p}\},$$
$$m_n = \min\{S_n, S_{n-1}, \ldots, S_{n-p}\}.$$

Taking into account the convexity relation (18), it is easy to see that the sequences $\{m_n\}$ and $\{M_n\}$ satisfy

$$0 \leqslant m_n \leqslant m_{n+1} \leqslant M_{n+1} \leqslant M_n \leqslant M_p, \qquad n \in \mathbb{N}.$$

Thus there always exist positive real numbers m, M, with $m \leqslant M$, such that

$$\lim_{n\to\infty} M_n = M \qquad \text{and} \qquad \lim_{n\to\infty} m_n = m.$$

Denoting $\delta_n = M_n - m_n$, one observes that S_n converges if and only if

$$\lim_{n\to\infty} \delta_n = 0. \tag{19}$$

We will finish the proof of Theorem 1 using some auxiliary lemmas, which will be shown below. In order to obtain the convergence of S_n, the following inequality will be proved (see Lemma 1)

$$\delta_n \leqslant (1 - t_n t_{n-1} \cdots t_{n-p+1})\, \delta_{n-p}, \qquad n \geqslant 2p, \quad \delta_p > 0. \tag{20}$$

Then (see Lemma 2), making use of the identity (13), it will be shown that condition (9) implies

$$\sum_n t_n t_{n-1} \cdots t_{n-p+1} = \infty,$$

thus obtaining the pointwise convergence of $\overrightarrow{S_n}(x)$ for each fixed x on $(-\infty, 0)$. ∎

Now we will present the auxiliary lemmas we used on the previous proof.

Lemma 1. *The inequality* (20) *holds for the sequences* $(\delta_n)_{n\geqslant p}$ *and* $(t_n)_{n\geqslant 1}$ *defined previously.*

Proof of Lemma 1. First, for the sake of clearness, we will prove the lemma for the particular case $p = 2$. The generalization to any p, which is totally analogous, will be given below. Let

$$X_n = \begin{pmatrix} S_n \\ S_{n-1} \\ S_{n-2} \end{pmatrix},$$

and put

$$X_n = \mathscr{W}_n X_{n-2},$$

where

$$\mathscr{W}_n = \begin{pmatrix} t_n t_{n-1} & (1-t_n) & t_n(1-t_{n-1}) \\ t_{n-1} & 0 & 1-t_{n-1} \\ 1 & 0 & 0 \end{pmatrix}. \tag{21}$$

We denote $\mathscr{W}_n = (w_n^{i,j})_{i,\, j \in \{1, 2, 3\}}$ and

$$M_n = \max\{S_n, S_{n-1}, S_{n-2}\},$$
$$m_n = \min\{S_n, S_{n-1}, S_{n-2}\}.$$

Writing $N_n = \{S_n, S_{n-1}, S_{n-2}\} \setminus \{M_n, m_n\}$ we have that

$$M_n = \alpha_n^0 M_{n-2} + \alpha_n^1 N_{n-2} + \alpha_n^2 m_{n-2},$$
$$m_n = \beta_n^0 M_{n-2} + \beta_n^1 N_{n-2} + \beta_n^2 m_{n-2},$$

where

$$\{\alpha_n^j\}_{(j=0,1,2)}, \{\beta_n^j\}_{(j=0,1,2)} \subset \{(w_n^{i,j})\}_{i,\, j \in \{0, 1, 2\}},$$

and

$$\sum_{j=0}^{2} \alpha_n^j = \sum_{j=0}^{2} \beta_n^j = 1.$$

Since $N_n \in [m_n, M_n]$, there exists ξ_n such that $0 < \xi_n < 1$ for $n > 2$ satisfying

$$N_n = \xi_n m_{n-2} + (1-\xi_n)\, M_{n-2}.$$

Thus we obtain

$$M_n = (1 - \alpha_n^1 \xi_n - \alpha_n^2)\, M_{n-2} + (\alpha_n^1 \xi_n + \alpha_n^2)\, m_{n-2},$$
$$m_n = (1 - \beta_n^1 \xi_n - \beta_n^2)\, M_{n-2} + (\beta_n^1 \xi_n + \beta_n^2)\, m_{n-2},$$

and we have

$$\delta_n = (1 - (\alpha_n^2 + \beta_n^0 + \alpha_n^1 \xi_n + \beta_n^1 (1-\xi_n)))\, \delta_{n-2}, \qquad n \geqslant 4, \delta_2 > 0.$$

Hence we obtain that

$$\delta_n \leqslant (1-\min(w_n^{i_0, j_0}+w_n^{i_1, j_1}+w_n^{i_2, j_2}))\, \delta_{n-2},$$

with $i_0 \neq i_1$, $j_0 \neq j_1 \neq j_2$, $i_2 = i_0$ or i_1. Taking into account inequality (17) and the distribution of the entries in the matrix (21), it is easy to see that the minimum on the right hand side of the previous inequality is reached for the following sum of entries of $\mathscr{W}_n$

$$w_n^{1,1}+w_n^{3,2}+w^{3,3} = t_n t_{n-1}.$$

Hence, we obtain the inequality

$$\delta_n \leqslant (1-t_n t_{n-1})\, \delta_{n-2}, \qquad n \geqslant 4,\ \delta_2 > 0,$$

which proves (20) for $p=2$.

Now we present the generalization to any p. Let

$$X_n = \begin{pmatrix} S_n \\ S_{n-1} \\ S_{n-2} \\ \vdots \\ S_{n-p} \end{pmatrix},$$

and put

$$X_n = \mathscr{W}_n X_{n-p},$$

where the square matrix $\mathscr{W}_n$ of order $p+1$ can be written, in a short way, as

$$\mathscr{W}_n = \begin{pmatrix} W_{1,n} & \mathscr{U}_n \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Here, the expression for $W_{1,n} \in \mathbb{R}^p$ is

$$W_{1,n} = \begin{pmatrix} t_n t_{n-1} \cdots t_{n-p+1} \\ t_{n-1} t_{n-2} \cdots t_{n-p+1} \\ \vdots \\ t_{n-j+1} t_{n-j} \cdots t_{n-p+1} \\ \vdots \\ t_{n-p+1} \end{pmatrix}, \qquad 1 \leqslant j \leqslant p,$$

and $\mathscr{U}_n$ is an upper triangular matrix of order p,

$$\mathscr{U}_n=\begin{pmatrix}
(1-t_n) & t_n(1-t_{n-1}) & \cdots & t_nt_{n-1}\cdots t_{n-j+2}(1-t_{n-j+1}) & \cdots & t_nt_{n-1}\cdots t_{n-p+2}(1-t_{n-p+1})\\
0 & 1-t_{n-1} & \cdots & t_{n-1}\cdots t_{n-j+2}(1-t_{n-j+1}) & \cdots & t_{n-1}\cdots t_{n-p+2}(1-t_{n-p+1})\\
\vdots & 0 & & \vdots & & \vdots\\
 & \vdots & \ddots & t_{n-j+2}(1-t_{n-j+1}) & & \\
 & & & 1-t_{n-j+1} & \cdots & t_{n-j+1}\cdots t_{n-p+2}(1-t_{n-p+1})\\
 & & & 0 & & \\
\vdots & & & \vdots & \ddots & \vdots\\
 & & & & & t_{n-p+2}(1-t_{n-p+1})\\
0 & 0 & \cdots & 0\ \cdots & 0 & 1-t_{n-p+1}
\end{pmatrix},$$

with $2 \leqslant j \leqslant p$. We denote $\mathscr{W}_n=(w_n^{i,j})_{i,j\in\{0,\dots,p\}}$ and as before

$$M_n=\max\{S_n,S_{n-1},S_{n-2},\dots,S_{n-p}\},\ \&m_n=\min\{S_n,S_{n-1},S_{n-2},\dots,S_{n-p}\}.$$

Writing $N_n^{(1)}, N_n^{(2)},\dots,N_n^{(p-1)}=\{S_n,S_{n-1},S_{n-2},\dots,S_{n-p}\}\setminus\{M_n,m_n\}$ we have that

$$M_n=\alpha_n^0M_{n-p}+\alpha_n^1N_{n-p}^{(1)}+\cdots+\alpha_n^{p-1}N_{n-p}^{(p-1)}+\alpha_n^pm_{n-p},$$
$$m_n=\beta_n^0M_{n-p}+\beta_n^1N_{n-p}^{(1)}+\cdots+\beta_n^{p-1}N_{n-p}^{(p-1)}+\beta_n^pm_{n-p},$$

where

$$\{\alpha_n^j\}_{(j=0,1,\dots,p)},\ \{\beta_n^j\}_{(j=0,1,\dots,p)}\subset\{(w_n^{i,j})\}_{i,j\in\{0,1,\dots,p\}},$$

and

$$\sum_{j=0}^{p}\alpha_n^j=\sum_{j=0}^{p}\beta_n^j=1. \tag{23}$$

Since $N_n^{(k)}\in[m_n,M_n]$, $k=1,\dots,p-1$, there exists $\xi_n^{(k)}$, $0<\xi_n<1$ for $n>p$, such that

$$N_n^{(k)}=\xi_n^{(k)}m_n+(1-\xi_n^{(k)})\,M_n,\qquad k=1,\dots,p-1.$$

Considering (23) we have

$$M_n=\left(1-\sum_{k=1}^{p-1}\alpha_n^k\xi_n^{(k)}-\alpha_n^p\right)M_{n-p}+\left(\sum_{k=1}^{p-1}\alpha_n^k\xi_n^{(k)}+\alpha_n^p\right)m_{n-p},$$
$$m_n=\left(1-\sum_{k=1}^{p-1}\beta_n^k\xi_n^{(k)}-\beta_n^p\right)M_{n-p}+\left(\sum_{k=1}^{p-1}\beta_n^k\xi_n^{(k)}+\beta_n^p\right)m_{n-p}.$$

Thus we obtain

$$\delta_n=\left(\sum_{k=1}^{p-1}(\beta_n^k-\alpha_n^k)\,\xi_n^{(k)}+\beta_n^p-\alpha_n^p\right)\delta_{n-p},\qquad n\geqslant 2p,\quad \delta_p>0.$$

Making use again of (23) we have that

$$\delta_n = \left(1 - (\alpha_n^p + \beta_n^0 + \sum_{k=1}^{p-1} (\alpha_n^k \xi_n^{(k)} + \beta_n^k (1 - \xi_n^{(k)})) \right) \delta_{n-p}.$$

Hence we obtain the inequality

$$\delta_n \leqslant (1 - \min(w_n^{i_0, j_0} + w_n^{i_1, j_1} + \cdots + w^{i_p, j_p}))\, \delta_{n-p}, \qquad n \geqslant 2p, \quad \delta_p > 0, \tag{24}$$

with $i_0 \neq i_1$, $j_0 \neq j_1 \neq \cdots \neq j_p$, $i_2, i_3, ..., i_p \in \{i_0, i_1\}$. As before, taking into account (17) and the distribution of the entries in the matrix (22), we verify that the minimum on the right hand side of the inequality (24) is reached for the following sum of entries of $\mathscr{W}_n$

$$w_n^{1,1} + w_n^{p+1,2} + w^{p+1,3} + \cdots + w^{p+1,p+1} = t_n t_{n-1} \cdots t_{n-p+1} > 0.$$

Hence, we obtain the inequality (20) and the lemma is proved. ∎

LEMMA 2. *The following holds:*

$$\sum_n \frac{1}{\sum_{j=0}^{p-1} \dfrac{1}{b_{n-j} b_{n-j-1} \cdots b_{n-j-p}}} = \infty \Rightarrow \sum_n t_n t_{n-1} \cdots t_{n-p+1} = \infty.$$

Proof of Lemma 2. Considering (16) we have that

$$t_n t_{n-1} \cdots t_{n-p+1} = b_n b_{n-1} \cdots b_{n-p+1} \frac{\Delta_{n-p}}{\Delta_n}. \tag{25}$$

From (15) it follows that

$$\Delta_n = b_n b_{n-1} \cdots b_{n-p+1} \Delta_{n-p} + \Delta_{n-p-1} + b_n \Delta_{n-p-2} + b_n b_{n-1} \Delta_{n-p-3} \\ + b_n b_{n-1} b_{n-2} \Delta_{n-p-4} + \cdots + b_n b_{n-1} \cdots b_{n-p+2} \Delta_{n-2p}.$$

Thus, from (25) it follows that

$$t_n t_{n-1} \cdots t_{n-p+1} = \frac{1}{1 + \dfrac{\Delta n - p - 1}{b_n b_{n-1} \cdots b_{n-p+1} \Delta_{n-p}} + \cdots + \dfrac{\Delta_{n-2p}}{b_{n-p+1} \Delta_{n-p}}}.$$

From (15) we get

$$\frac{\Delta_{n-p-j}}{\Delta_{n-p}} \leqslant \frac{1}{b_{n-p} \cdots b_{n-p-j+1}}, \qquad 1 \leqslant j \leqslant p.$$

Hence we can conclude that

$$t_n t_{n-1} \cdots t_{n-p+1} \geqslant \frac{1}{1 + \sum_{j=0}^{p-1} \frac{1}{b_{n-j} b_{n-j-1} \cdots b_{n-j-p}}},$$

and the lemma is proved. ∎

Remark 1. We point out that for $p=1$ the sufficient condition in Theorem 1 turns out to be the known condition of the scalar casementioned before:

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty.$$

Remark 2. The divergence of this series also implies the convergence of the vector continued fraction when the coefficients a_n have a "regular behaviour," for instance when the sequence a_n is monotone or has a bounded rate of increase. There is an important fact that can be concluded from Theorem 1. If the entries of the non-bounded operator (6) satisfy the hypothesis of Theorem 1, then this operator has a unique set of spectral data (the set of resolvent functions) which defines the operator in a unique way. These last circumstances give a connection between the vector case and the determinacy of the moment problem.

4. GENERALIZED STIELTJES CONVERGENCE CONDITION

Considering the equivalent form (12) of the Stieltjes continued fraction (7), one finds that the classical Stieltjes convergence condition (2) remains true in the vector case, as stated in the following theorem:

THEOREM 2. *For the vector Stieltjes continued fraction* (12) *one has*

$$\vec{S_n}(z) \underset{n \to \infty}{\rightrightarrows} \vec{S}(z) \Rightarrow \sum_{k=1}^{\infty} |b_k| = \infty, \qquad z \in K \subseteq \mathbb{C} \setminus [0, \infty),$$

where

$$b_k < 0 \ \textit{for} \ k \equiv 1 \ (\mathrm{mod}\ p+1), \qquad b_k > 0 \ \textit{otherwise}.$$

Before proving Theorem 2 we will present some auxiliary results. We use the same notation as in the proof of Theorem 1. We fix $x \in (-\infty, 0)$ and write $b_n = b_n(x)$, so that $b_n > 0$ for every $n \in \mathbb{N}$, see (14). Analogously, let $S_n = S_n^1(x)$. Hence, we will prove that if S_n converges then

$$\sum_n b_n = \infty. \tag{26}$$

We have the following lemmas:

LEMMA 3. *For the sequences* $(b_n)_{n \geqslant 1}$ *and* $(t_n)_{n \geqslant 1}$, *previously defined by* (14) *and* (16) *respectively, one has*

$$\sum_n b_n = \infty \Leftrightarrow \sum_n t_n = \infty.$$

Lemma 3 is a direct consequence of the following Lemma 4:

LEMMA 4. *Let* $(b_n)_{n \geqslant 1}$ *be a sequence of positive real numbers and* $(\Delta_n)_{n \geqslant 0}$ *a sequence defined by the recurrence* (15) *with the same initial conditions. Then*

$$\sum_n b_n = \infty \Leftrightarrow \sum_n \left(1 - \frac{\Delta_{n-p-1}}{\Delta_n} \frac{\Delta_{n-p-2}}{\Delta_{n-1}}\right) = \infty.$$

Proof of Lemma 4. First we will prove that

$$\sum_n b_n = \infty \Rightarrow \sum_n \left(1 - \frac{\Delta_{n-p-1}}{\Delta_n} \frac{\Delta_{n-p-2}}{\Delta_{n-1}}\right) = \infty.$$

Taking into account (15), we have

$$\Delta_n > \Delta_{n-p-1}, \tag{27}$$

which implies that

$$\Delta_n > \min\{\Delta_0, \Delta_1, \ldots, \Delta_p\} > 0,$$

from which we obtain

$$\sum_n b_n \Delta_{n-1}^2 = \infty \qquad \text{and} \qquad \sum_n b_{n-1} \Delta_{n-2} \Delta_{n-p-1} = \infty.$$

We will prove that there exists $\Lambda_1 \subset \mathbb{N}$ such that

$$\Delta_{n_k} \Delta_{n_{k-1}} \xrightarrow[n_k \in \Lambda_1]{} \infty. \tag{28}$$

From (15) it follows that

$$\Delta_n \Delta_{n-1} = b_n \Delta_{n-1}^2 + b_{n-1} \Delta_{n-2} \Delta_{n-p-1} + \Delta_{n-p-2} \Delta_{n-p-1}.$$

Thus

$$\Delta_n \Delta_{n-1} - \Delta_{n-p-2} \Delta_{n-p-1} = b_n \Delta_{n-1}^2 + b_{n-1} \Delta_{n-2} \Delta_{n-p-1},$$

which implies

$$\sum_{n=1}^{N} (\Delta_n \Delta_{n-1} - \Delta_{n-p-2} \Delta_{n-p-1}) = \sum_{n=1}^{N} (b_n \Delta_{n-1}^2 + b_{n-1} \Delta_{n-2} \Delta_{n-p-1}).$$

We have

$$\sum_{j=0}^{p} \Delta_{n-j} \Delta_{n-j-1} \xrightarrow[n]{} \infty,$$

hence there exists $\Lambda_1 \subset \mathbb{N}$ such that (28) holds. Now we can suppose that

$$\frac{\Delta_{n-p-1}}{\Delta_n} \frac{\Delta_{n-p-2}}{\Delta_{n-1}} \xrightarrow[n]{} 1, \tag{29}$$

otherwise the proof would stop right here. From (29) we have that there exists $N_0 \in \mathbb{N}$ such that

$$1 > \frac{\Delta_{n-p-1}}{\Delta_n} \frac{\Delta_{n-p-2}}{\Delta_{n-1}} > \frac{1}{2}, \qquad n \geqslant N_0.$$

Thus we obtain the inequalities

$$\begin{aligned} 1 - \frac{\Delta_{n-p-1}}{\Delta_n} \frac{\Delta_{n-p-2}}{\Delta_{n-1}} &> \frac{1}{2} \left(\frac{\Delta_n \Delta_{n-1}}{\Delta_{n-p-1} \Delta_{n-p-2}} - 1 \right) \\ &> \frac{1}{2} \ln \frac{\Delta_n \Delta_{n-1}}{\Delta_{n-p-1} \Delta_{n-p-2}}. \end{aligned}$$

Taking the sum we obtain

$$\sum_{n=N_0}^{N}\left(1-\frac{\Delta_{n-p-1}}{\Delta_n}\,\frac{\Delta_{n-p-2}}{\Delta_{n-1}}\right)>\frac{1}{2}\sum_{n=N_0}^{N}\ln\frac{\Delta_n\Delta_{n-1}}{\Delta_{n-p-1}\Delta_{n-p-2}}$$
$$>\frac{1}{2}\ln\frac{\Delta_N\Delta_{N-1}^2\Delta_{N-2}^2\cdots\Delta_{N-p}^2\Delta_{N-p+1}}{\Delta_{N_0-1}\Delta_{N_0-2}^2\cdots\Delta_{N_0-p-1}^2\Delta_{N_0-p-2}}.$$

Considering (28) and taking into account that $\Delta_n>c>0$ for all $n\in\mathbb{N}$, we have

$$\Delta_{n_k}\Delta_{n_k-1}^2\cdots\Delta_{n_k-p}^2\Delta_{n_k-p-1}\xrightarrow[n_k\in\Lambda_1]{}\infty.$$

Hence

$$\sum_{n=1}^{\infty}\left(1-\frac{\Delta_{n-p-1}}{\Delta_n}\,\frac{\Delta_{n-p-2}}{\Delta_{n-1}}\right)=\infty.$$

In order to prove the other implication

$$\sum_n\left(1-\frac{\Delta_{n-p-1}}{\Delta_n}\,\frac{\Delta_{n-p-2}}{\Delta_{n-1}}\right)=\infty\Rightarrow\sum_n b_n=\infty,$$

we will prove that

$$\Delta_n+\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p}\xrightarrow[n]{}\infty, \tag{30}$$

because one always has

$$b_n>\frac{b_n\Delta_{n-1}}{\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p}+\Delta_{n-p-1}}$$
$$=\frac{\Delta_n+\Delta_{n-1}+\cdots+\Delta_{n-p}}{\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p}+\Delta_{n-p-1}}-1,$$

and since $\Delta_n>\Delta_{n-p-1}$ one has

$$\frac{\Delta_n+\Delta_{n-1}+\cdots+\Delta_{n-p}}{\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p}+\Delta_{n-p-1}}>1.$$

Thus

$$\frac{\Delta_n+\Delta_{n-1}+\cdots+\Delta_{n-p}}{\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p}+\Delta_{n-p-1}}-1>\ln\frac{\Delta_n+\Delta_{n-1}+\cdots+\Delta_{n-p}}{\Delta_{n-1}+\Delta_{n-2}+\cdots+\Delta_{n-p-1}},$$

and we obtain

$$\sum_{n=p-1}^{N} b_n > \ln \frac{\Delta_N + \Delta_{N-1} + \cdots + \Delta_{N-p}}{\Delta_p + \Delta_{p-1} + \cdots + \Delta_0}.$$

If (30) holds, then the previous inequality would imply the divergence of the series on the left hand side. Hence, to conclude the proof of this lemma it only remains to check that (30) holds. We have that

$$1 - \frac{\Delta_{n-p-1}\Delta_{n-p-2}}{\Delta_n \Delta_{n-1}} < \ln \frac{\Delta_n \Delta_{n-1}}{\Delta_{n-p-1}\Delta_{n-p-2}},$$

and therefore

$$\begin{aligned}\sum_{n=p+2}^{N} \left(1 - \frac{\Delta_{n-p-1}\Delta_{n-p-2}}{\Delta_n \Delta_{n-1}}\right) &< \sum_{n=p+2}^{N} \ln \frac{\Delta_n \Delta_{n-1}}{\Delta_{n-p-1}\Delta_{n-p-2}} \\ &= \ln \frac{\Delta_N \Delta_{N-1}^2 \cdots \Delta_{N-p}^2 \Delta_{N-p-1}}{\Delta_{p-1}\Delta_{p-2}^2 \cdots \Delta_1^2 \Delta_0}.\end{aligned}$$

Hence

$$\Delta_N \Delta_{N-1}^2 \cdots \Delta_{N-p}^2 \Delta_{N-p-1} \xrightarrow[N]{} \infty,$$

and taking (27) into account, one finds

$$\Delta_N \Delta_{N-1} \cdots \Delta_{N-p} \xrightarrow[N]{} \infty,$$

and we obtain (30). ∎

Proof of Lemma 3. Making use of the recurrence (15), we have

$$\begin{aligned}1 - \frac{\Delta_{n-p-1}\Delta_{n-p-2}}{\Delta_n \Delta_{n-1}} &= \frac{\Delta_n(\Delta_{n-1} - \Delta_{n-p-2}) + \Delta_{n-p-2}(\Delta_n - \Delta_{n-p-1})}{\Delta_n \Delta_{n-1}} \\ &= b_{n-1} \frac{\Delta_{n-2}}{\Delta_{n-1}} + b_n \frac{\Delta_{n-p-2}}{\Delta_n} \\ &= t_{n-1} + t_n(1 - t_{n-1}).\end{aligned}$$

From (17) one finds

$$\sum_n t_n = \infty \Leftrightarrow \sum_n (t_{n-1} + t_n(1 - t_{n-1})) = \infty,$$

and thus, from Lemma 4, we have that the present lemma is true. ∎

Now we are ready to prove the theorem.

Proof of Theorem 2. We will prove the inequality:

$$\delta_{n+1} \geqslant (1-t_{n+1})\, \delta_n, \qquad \forall n \in \mathbb{N}. \tag{31}$$

From the proof of Theorem 1 we know that if S_n converges then (19) holds. Thus we would have the implication

$$\lim_{n \to \infty} \delta_n = 0 \Rightarrow \sum_n t_n = \infty,$$

and from Lemma 3 we would obtain (26). To prove (31) we have to take into account the set of indices on which δ_n and δ_{n+1} depend, respectively. We can write

$$\delta_n \sim \begin{pmatrix} n \\ n-1 \\ \vdots \\ n-p+1 \\ n-p \end{pmatrix} \qquad \text{and} \qquad \delta_{n+1} \sim \begin{pmatrix} n+1 \\ n \\ \vdots \\ n-p+2 \\ n-p+1 \end{pmatrix}.$$

Now, we will differentiate between the following cases:

Case 1. There exist j, k with $0 \leqslant j, k \leqslant p-1$, $j \neq k$, such that

$$\delta_n = |S_{n-j} - S_{n-k}|.$$

Notice that this case only holds for $p > 1$. We have that $S_{n-p} \in (S_{n-j}, S_{n-k})$. Since $\{S_n, ..., S_{n-p}\} \subset [S_{n-j}, S_{n-k}]$, considering without loss of generality $S_n < S_{n-p}$ and making use of (18) we have

$$S_{n+1} \in [S_n, S_{n-p}] \subset [S_{n-j}, S_{n-k}].$$

Thus, we obtain

$$\delta_{n+1} = \delta_n > (1-t_{n+1})\, \delta_n.$$

Case 2. There exists j with $0 \leqslant j \leqslant p-1$, such that

$$\delta_n = |S_{n-p} - S_{n-j}|. \tag{32}$$

Thus

$$\delta_{n+1} = |S_{n-k} - S_{n-l}|,$$

where $-1 \leqslant k, l \leqslant p-1, l \neq k$. Without loss of generality we can suppose $S_{n-p} > S_{n-j}$, where S_{n-j} is such that (32) holds. Making use of (18) we have that

$$S_{n+1} \in [S_n, S_{n-p}] \subset [S_{n-j}, S_{n-p}],$$

and hence

$$\{S_{n-p+1}, \ldots, S_n, S_{n+1}\} \subset [S_{n-j}, S_{n-p}].$$

Thus

$$\delta_{n+1} = |S_{n-k} - S_{n-j}|$$

where S_{n-j} is the one from (32) and k is such that $-1 \leqslant k \leqslant p-1, k \neq j$. We have the following situation:

$$S_{n-j} \leqslant S_n < S_{n+1} \leqslant S_{n-k} < S_{n-p}.$$

One has

$$\delta_{n+1} = S_{n-k} - S_{n-j} = S_{n-p} - S_{n-j} - (S_{n-p} - S_{n-k}),$$

hence

$$\delta_{n+1} = \delta_n + (S_{n-k} - S_{n+1}) - (S_{n-p} - S_{n+1}),$$

and we obtain

$$\delta_{n+1} \geqslant \delta_n - (S_{n-p} - S_{n+1}). \tag{33}$$

Making use of (18) for S_{n+1}, we have that

$$S_{n+1} - S_{n-p} = t_{n+1}(S_n - S_{n-p}).$$

Thus

$$|S_{n+1} - S_{n-p}| \leqslant t_{n+1}\delta_n.$$

Considering (33) we have

$$\delta_{n+1} \geqslant \delta_n(1 - t_{n+1}).$$

Having proved (31), we conclude that Theorem 2 holds. ∎

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